

Distinct distances on two lines*

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February 14, 2013

Dedicated to the memory of Gyuri Elekes, for having shown us the light.

Abstract

Let \mathcal{P}_1 and \mathcal{P}_2 be two finite sets of points in the plane, so that \mathcal{P}_1 is contained in a line ℓ_1 , \mathcal{P}_2 is contained in a line ℓ_2 , and ℓ_1 and ℓ_2 are neither parallel nor orthogonal. Then the number of distinct distances determined by the pairs of $\mathcal{P}_1 \times \mathcal{P}_2$ is

$$\Omega\left(\min\left\{|\mathcal{P}_1|^{2/3}|\mathcal{P}_2|^{2/3}, |\mathcal{P}_1|^2, |\mathcal{P}_2|^2\right\}\right).$$

In particular, if $|\mathcal{P}_1| = |\mathcal{P}_2| = m$, then the number of these distinct distances is $\Omega(m^{4/3})$, improving upon the previous bound $\Omega(m^{5/4})$ of Elekes [4].

1 Introduction

Given a set \mathcal{P} of m points in \mathbb{R}^2 , let $D(\mathcal{P})$ denote the number of distinct distances that are determined by pairs of points from \mathcal{P} . Let $D(m) = \min_{|\mathcal{P}|=m} D(\mathcal{P})$; that is, $D(m)$ is the minimum number of distinct distances that any set of m points in \mathbb{R}^2 must always determine. In his celebrated 1946 paper [7], Erdős derived the bound $D(m) = O(m/\sqrt{\log m})$. For the celebrations of his 80'th birthday, Erdős compiled a survey of his favorite contributions to mathematics [8], in which he wrote “My most striking contribution to geometry is, no doubt, my problem on the number of distinct distances. This can be found in many of my papers on combinatorial and geometric problems”. Recently, after 65 years and a series of increasingly larger lower bounds, Guth and Katz [9] provided an almost matching lower bound $D(m) = \Omega(m/\log m)$. For this, Guth and Katz developed several novel techniques, relying on tools from algebraic geometry.

While the problem of obtaining the asymptotic value of $D(m)$ is almost settled, many other variants of the distinct distances problem are still widely open. For example, see [3, 13] regarding the conjecture that any m points in convex position in the plane determine at least $\lfloor m/2 \rfloor$ distinct distances, and [15] for a study of the minimum number of distinct distances in higher dimensions.

*Work on this paper was partially supported by Grant 338/09 from the Israel Science Fund and by the Israeli Centers of Research Excellence (I-CORE) program (Center No. 4/11). Work by Micha Sharir was also supported by NSF Grant CCF-08-30272 and by the Hermann Minkowski-MINERVA Center for Geometry at Tel Aviv University.

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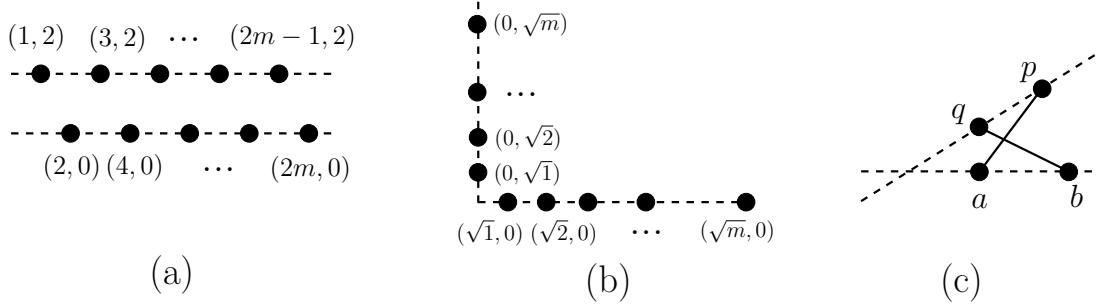


Figure 1: (a) Two parallel lines with $D(\mathcal{P}_1, \mathcal{P}_2) = \Theta(m)$. (b) Two orthogonal lines with $D(\mathcal{P}_1, \mathcal{P}_2) = \Theta(m)$. (c) A quadruple (a, p, b, q) in Q .

In this paper we consider the following variant of the distinct distances problem in the plane. Let \mathcal{P}_1 and \mathcal{P}_2 be two sets of m points each, such that all the points of \mathcal{P}_1 (resp., \mathcal{P}_2) lie on a line ℓ_1 (resp., ℓ_2). Let $D(\mathcal{P}_1, \mathcal{P}_2)$ denote the number of distinct distances between the pairs of $\mathcal{P}_1 \times \mathcal{P}_2$. When the two lines are either parallel or orthogonal, the points can be arranged such that $D(\mathcal{P}_1, \mathcal{P}_2) = \Theta(m)$; for example, see Figures 1(a,b). Purdy conjectured that if the lines are neither parallel nor orthogonal then $D(\mathcal{P}_1, \mathcal{P}_2) = \omega(m)$ (e.g., see [1, Section 5.5]). Elekes and Rónyai [5] proved that the number of distinct distances in such a scenario is indeed superlinear, though without supplying a specific superlinear lower bound. Elekes [4] derived the improved bound $D(\mathcal{P}_1, \mathcal{P}_2) = \Omega(m^{5/4})$ (when the lines are neither parallel nor orthogonal) and gave a construction with $D(\mathcal{P}_1, \mathcal{P}_2) = O(m^2/(\log m)^{1/2})$, in which the angle between the two lines is $\pi/3$. Previously, these were the best known bounds for $D(\mathcal{P}_1, \mathcal{P}_2)$. Schwartz, Solymosi, and de Zeeuw [14] have recently shown, among several other related results, that the number of distinct distances remains superlinear when $|\mathcal{P}_1| = m^{1/2+\varepsilon}$ and $|\mathcal{P}_2| = m$, for any $\varepsilon > 0$.

In this paper we derive the following result, for point sets $\mathcal{P}_1, \mathcal{P}_2$ of arbitrary (possibly different) cardinalities.

Theorem 1.1 *Let \mathcal{P}_1 and \mathcal{P}_2 be two sets of points in \mathbb{R}^2 of cardinalities n and m , respectively, such that the points of \mathcal{P}_1 all lie on a line ℓ_1 , the points of \mathcal{P}_2 all lie on a line ℓ_2 , and the two lines are neither parallel nor orthogonal. Then the number of distinct distances determined by the pairs in $\mathcal{P}_1 \times \mathcal{P}_2$ is*

$$\Omega\left(\min\left\{n^{2/3}m^{2/3}, n^2, m^2\right\}\right).$$

Theorem 1.1 immediately implies the following improved lower bound for the “balanced” case.

Corollary 1.2 *Let \mathcal{P}_1 and \mathcal{P}_2 be two sets of points in \mathbb{R}^2 , each of cardinality m , such that the points of \mathcal{P}_1 all lie on a line ℓ_1 , the points of \mathcal{P}_2 all lie on a line ℓ_2 , and the two lines are neither parallel nor orthogonal. Then there are $\Omega(m^{4/3})$ distinct distances between the pairs in $\mathcal{P}_1 \times \mathcal{P}_2$.*

Even with this improved lower bound (over the lower bound in [4]), there is still a considerable gap to the near-quadratic upper bound in [4], and the prevailing belief is that the correct lower bound is indeed close to quadratic.

To obtain the improved bound, we use a variant of the reduction devised by Elekes and presented in Elekes and Sharir [6]. Unlike Guth and Katz [9], who refined this reduction

to prove their bound for the general planar distinct distances problem, our proof does not require the full power of the new algebraic techniques. This is mainly because our modified reduction transforms the problem into an incidence problem in a parametric *plane*, rather than in 3-space (as is the case in [9]), thereby eliminating the need for most of the sophistication in the analysis in [9].

2 The proof of Theorem 1.1

Without loss of generality, we assume that the points of \mathcal{P}_1 are on one side of the point $\ell_1 \cap \ell_2$. Otherwise, we can partition \mathcal{P}_1 into two subsets by splitting ℓ_1 at $\ell_1 \cap \ell_2$, and remove the subset that yields fewer distinct distances with the points of \mathcal{P}_2 . At worst, this halves the number of distinct distances between the pairs in $\mathcal{P}_1 \times \mathcal{P}_2$. For the same reason, we may assume that the points of \mathcal{P}_2 are also on one side of $\ell_1 \cap \ell_2$. Furthermore, without loss of generality, we assume that $n = |\mathcal{P}_1| \geq m = |\mathcal{P}_2|$.

We begin with a variant of the reduction from [6]. We set $x = D(\mathcal{P}_1, \mathcal{P}_2)$ and denote the x distinct distances in $\mathcal{P}_1 \times \mathcal{P}_2$ as $\delta_1, \dots, \delta_x$. For a pair of points u and v , we denote by $|uv|$ the (Euclidean) length of the straight segment uv . Let Q be the set of quadruples (a, p, b, q) where $a, b \in \mathcal{P}_1$ and $p, q \in \mathcal{P}_2$, such that $|ap| = |bq| > 0$ and $ap \neq bq$ (the two segments are allowed to share at most one endpoint); see Figure 1(c). The quadruples are ordered, so that (a, p, b, q) and (b, q, a, p) are considered as two distinct elements of Q .

Let $E_i = \{(a, p) \in \mathcal{P}_1 \times \mathcal{P}_2 \mid |ap| = \delta_i\}$, for $i = 1, \dots, x$. We have, by the Cauchy-Schwarz inequality,

$$|Q| = 2 \sum_{i=1}^x \binom{|E_i|}{2} \geq \sum_{i=1}^x (|E_i| - 1)^2 \geq \frac{1}{x} \left(\sum_{i=1}^x (|E_i| - 1) \right)^2 = \frac{(mn - x)^2}{x}. \quad (1)$$

In the remainder of the proof we derive an upper bound on $|Q|$, showing that $|Q| = O(m^{4/3}n^{4/3} + n^2)$. This, combined with (1), yields the lower bound asserted in the theorem. To establish this upper bound, consider the set \mathcal{R} of all rotations of the plane, where each rotation is around some point u by some angle α .¹ Given a quadruple $(a, p, b, q) \in Q$, there is a unique rotation of this form (with a unique pair (u, α)) that maps a to b and p to q . We say that this is the rotation that corresponds to the quadruple (a, p, b, q) .

Lemma 2.1 *Every rotation corresponds to at most one quadruple of Q .*

Proof. Let (u, α) represent a rotation ρ that corresponds to the quadruple (a, p, b, q) . Without loss of generality, assume that $a \neq b$. Then u lies on the perpendicular bisector of ab . If u is not the midpoint of ab then, as is easily checked, a is the only point on ℓ_1 that ρ maps to another point on ℓ_1 (namely, to b). In this case, any other quadruple that ρ corresponds to must share the points a and b with the first quadruple. A symmetric argument applies to p and q , and implies that ρ can correspond only to (a, p, b, q) , unless either (i) $p = q$, or (ii) the center u lies on ℓ_1 or on ℓ_2 . In the former case, $u = p = q$ and $\alpha \neq \pi$, which is easily seen to imply the uniqueness of the quadruple. In the latter cases, assume, without loss of generality, that u lies on ℓ_1 , at the midpoint of ab . Then we must have $\alpha = \pi$, and ℓ_2 is mapped to its reflection about u . Since u is different from the point

¹In general, \mathcal{R} also includes translations, regarded as rotations around points at infinity, but these will not arise in our context.

$\ell_1 \cap \ell_2$ (this follows since all points of \mathcal{P}_1 lie on one side of $\ell_1 \cap \ell_2$), the reflection of ℓ_2 is disjoint from ℓ_2 , so no point on ℓ_2 is mapped to another point on ℓ_2 . \square

By Lemma 2.1, $|Q|$ is the number of rotations that take a point of \mathcal{P}_1 to a point of \mathcal{P}_1 and a point of \mathcal{P}_2 to a point of \mathcal{P}_2 .

Our next task is to find an algebraic characterization of the quadruples of Q . We first dispose of quadruples whose points are not all distinct. Consider such a quadruple (a, p, a, q) , say, and notice that the corresponding rotation is around a . Moreover, a must be incident to the perpendicular bisector of pq . Every point $p \in \mathcal{P}_2$ can participate in at most one pair (p, q) whose bisector is incident to a , so there are at most m quadruples whose corresponding rotation is around a . The same argument holds symmetrically for rotations around a point of \mathcal{P}_2 . Overall, the number of quadruples in Q whose corresponding rotation is around a point in \mathcal{P}_1 or in \mathcal{P}_2 is $O(mn)$, which is significantly smaller than the bound promised above for the number of quadruples of Q . Thus, in the following analysis, we only consider rotations that are not around a point of \mathcal{P}_1 or \mathcal{P}_2 , that is, corresponding to quadruples consisting of four distinct points.

Following the transformation in Guth and Katz [9] (inspired by a similar transformation in [6]), a rotation around the point (x, y) at an angle of α is mapped to the point $(x, y, \cot(\alpha/2)) \in \mathbb{R}^3$. With this mapping, given a pair of points $p = (p_x, p_y)$ and $q = (q_x, q_y)$ in the plane, the set of rotations that take p to q is mapped to a line in \mathbb{R}^3 , denoted as ℓ_{pq} and represented parametrically as

$$\left(\frac{p_x + q_x}{2}, \frac{p_y + q_y}{2}, 0 \right) + t \left(\frac{q_y - p_y}{2}, \frac{p_x - q_x}{2}, 1 \right), \quad \text{for } t \in \mathbb{R}. \quad (2)$$

We denote by \mathcal{L}_1 (resp., \mathcal{L}_2) the set of $2\binom{n}{2}$ lines (resp., $2\binom{m}{2}$ lines) in \mathbb{R}^3 , each corresponding to a different ordered pair of distinct points of \mathcal{P}_1 (resp., of \mathcal{P}_2). We refer to the lines of \mathcal{L}_1 as *blue lines* and to those of \mathcal{L}_2 as *red lines*.

We rotate, translate, and possibly reflect the original plane, so that the origin o is $\ell_1 \cap \ell_2$, ℓ_1 is the x -axis, the points of \mathcal{P}_1 lie on the positive side of o , and the points of \mathcal{P}_2 lie above ℓ_1 . We denote the slope of ℓ_2 as $s \neq 0, \infty$. As already observed, given a pair of points a, b , every rotation that takes a to b is around a point on the perpendicular bisector of ab . The line ℓ_{ab} in \mathbb{R}^3 that corresponds to the rotations that take a to b is a suitable “lifting” of the perpendicular bisector of ab . If a and b are both incident to ℓ_1 , the perpendicular bisector is a vertical line (i.e., parallel to the y -axis). Thus, the projection of a blue line on the xy -plane is parallel to the y -axis. Similarly, the projections of the red lines on the xy -plane all have slope $-1/s$.

Every intersection in \mathbb{R}^3 of a blue line ℓ_{ab} , for $a, b \in \mathcal{P}_1$, with a red line ℓ_{pq} , for $p, q \in \mathcal{P}_2$, corresponds in a 1-1 manner to a rotation that takes a to b and p to q . Thus, $|Q|$ is the number of red-blue intersections between these lines. Consider such a pair of intersecting lines ℓ_{ab} and ℓ_{pq} . Denote the coordinates of a and b as $(a_x, 0)$ and $(b_x, 0)$, respectively, and the coordinates of p and q as (p_x, sp_x) and (q_x, sq_x) , respectively. To characterize the property that (a, p, b, q) is in Q (that is, that ℓ_{ab} and ℓ_{pq} intersect), we set $\phi_{ab} = (a_x + b_x)/2$ and $\mu_{ab} = (a_x - b_x)/2$. By (2), the blue line that corresponds to the pair $\{a, b\}$ is parameterized as

$$\ell_{ab} = (\phi_{ab}, 0, 0) + t(0, \mu_{ab}, 1), \quad \text{for } t \in \mathbb{R}.$$

Similarly, the red line that corresponds to the pair $\{p, q\}$ is parameterized as

$$\ell_{pq} = (\phi_{pq}, s\phi_{pq}, 0) + t(-s\mu_{pq}, \mu_{pq}, 1), \quad \text{for } t \in \mathbb{R},$$

where $\phi_{pq} = (p_x + q_x)/2$ and $\mu_{pq} = (p_x - q_x)/2$. Denote the anticipated intersection point of ℓ_{ab} and ℓ_{pq} as u , and note that for the two lines to have the same z -coordinate at u , they must have the same value of t at u . Thus, for the intersection to exist, we must have

$$\phi_{ab} = \phi_{pq} - s\mu_{pq}t \quad \text{and} \quad t\mu_{ab} = s\phi_{pq} + \mu_{pq}t.$$

Eliminating t from this system yields

$$\frac{\phi_{pq} - \phi_{ab}}{s\mu_{pq}} = \frac{s\phi_{pq}}{\mu_{ab} - \mu_{pq}},$$

or

$$(\phi_{pq} - \phi_{ab})(\mu_{ab} - \mu_{pq}) = s^2\phi_{pq}\mu_{pq}. \quad (3)$$

Consider a pair of distinct points $p = (p_x, sp_x)$ and $q = (q_x, sq_x)$, both in \mathcal{P}_2 . We say that the point (ϕ_{pq}, μ_{pq}) corresponds to the pair $\{p, q\}$ and denote by \mathcal{V}_2 the set of all points that correspond to pairs from \mathcal{P}_2 . It is easily checked that no point of \mathcal{V}_2 can correspond to two distinct pairs. Next, consider a pair of distinct points $a = (a_x, 0)$ and $b = (b_x, 0)$, both in \mathcal{P}_1 , and define a curve $\gamma_{a,b}$, corresponding to the pair (a, b) , as

$$\gamma_{a,b} := (x - \phi_{ab})(\mu_{ab} - y) - s^2xy = 0. \quad (4)$$

The curve $\gamma_{a,b}$ is a non-degenerate hyperbola; since $s \neq 0, \infty$, it does not have non-trivial (that is, linear) factors, as is easily checked. Let \mathcal{C}_1 denote the resulting set of all these hyperbolas that correspond to pairs of points from \mathcal{P}_1 . By (3), every quadruple $(a, p, b, q) \in Q$ corresponds, in a 1-1 manner, to an incidence between the point (ϕ_{pq}, μ_{pq}) of \mathcal{V}_2 and the hyperbola $\gamma_{a,b}$ of \mathcal{C}_1 . Thus, $|Q| = I(\mathcal{V}_2, \mathcal{C}_1)$ (that is, the number of incidences between \mathcal{V}_2 and \mathcal{C}_1).

Lemma 2.2 *Let (a, b) and (a', b') be two distinct pairs in $\mathcal{P}_1 \times \mathcal{P}_1$. Then $\gamma_{a,b}$ and $\gamma_{a',b'}$ have at most two common points.*

Proof. Consider a common point (ϕ_{pq}, μ_{pq}) of $\gamma_{a,b}$ and $\gamma_{a',b'}$. By (4), we have

$$(\phi_{pq} - \phi_{ab})(\mu_{ab} - \mu_{pq}) = s^2\phi_{pq}\mu_{pq} \quad \text{and} \quad (\phi_{pq} - \phi_{a'b'})(\mu_{a'b'} - \mu_{pq}) = s^2\phi_{pq}\mu_{pq}, \quad (5)$$

or

$$\phi_{pq}\mu_{ab} - \phi_{ab}\mu_{pq} + \phi_{ab}\mu_{pq} = \phi_{pq}\mu_{a'b'} - \phi_{a'b'}\mu_{a'b'} + \phi_{a'b'}\mu_{pq}. \quad (6)$$

For now, we assume that $\phi_{ab} \neq \phi_{a'b'}$ and $\mu_{ab} \neq \mu_{a'b'}$. In this case, we get

$$\mu_{pq} = \frac{\phi_{pq}(\mu_{a'b'} - \mu_{ab}) + \phi_{ab}\mu_{ab} - \phi_{a'b'}\mu_{a'b'}}{\phi_{ab} - \phi_{a'b'}}.$$

Combining this with the first part of (5) implies

$$\begin{aligned} (\phi_{pq} - \phi_{ab}) \left(\mu_{ab} - \frac{\phi_{pq}(\mu_{a'b'} - \mu_{ab}) + \phi_{ab}\mu_{ab} - \phi_{a'b'}\mu_{a'b'}}{\phi_{ab} - \phi_{a'b'}} \right) \\ = s^2\phi_{pq} \frac{\phi_{pq}(\mu_{a'b'} - \mu_{ab}) + \phi_{ab}\mu_{ab} - \phi_{a'b'}\mu_{a'b'}}{\phi_{ab} - \phi_{a'b'}}. \end{aligned}$$

This is a quadratic equation in ϕ_{pq} , where the coefficient of ϕ_{pq}^2 is

$$(1 + s^2) \frac{\mu_{a'b'} - \mu_{ab}}{\phi_{ab} - \phi_{a'b'}}.$$

By the above assumption, this coefficient is not 0, and thus the equation has at most two solutions.

We next consider the cases where either $\phi_{ab} = \phi_{a'b'}$ or $\mu_{ab} = \mu_{a'b'}$ (both equalities cannot hold simultaneously, since $(a, b) \neq (a', b')$). When $\mu_{ab} = \mu_{a'b'}$, (6) becomes

$$-\phi_{ab}\mu_{ab} + \phi_{ab}\mu_{pq} = -\phi_{a'b'}\mu_{ab} + \phi_{a'b'}\mu_{pq},$$

which implies the unique solution

$$\mu_{pq} = \frac{\mu_{ab}(\phi_{ab} - \phi_{a'b'})}{\phi_{ab} - \phi_{a'b'}} = \mu_{ab} \quad \text{and} \quad \phi_{pq} = 0.$$

Similarly, when $\phi_{ab} = \phi_{a'b'}$ holds, (6) becomes

$$\phi_{pq}\mu_{ab} - \phi_{ab}\mu_{ab} = \phi_{pq}\mu_{a'b'} - \phi_{ab}\mu_{a'b'},$$

which implies the unique solution

$$\phi_{pq} = \frac{\phi_{ab}(\mu_{ab} - \mu_{a'b'})}{\mu_{ab} - \mu_{a'b'}} = \phi_{ab} \quad \text{and} \quad \mu_{pq} = 0.$$

As a matter of fact, the last two cases are impossible: $\phi_{pq} = (p_x + q_x)/2$ cannot be 0 by our assumption that all points of \mathcal{P}_2 lie on one side of o (and that ℓ_2 is not vertical), and $\mu_{pq} = (p_x - q_x)/2$ also cannot be 0 because we assume here that $p \neq q$. In conclusion, $\gamma_{a,b}$ and $\gamma_{a',b'}$ have at most two common points. \square

Corollary 2.3 *The incidence graph in $\mathcal{V}_2 \times \mathcal{C}_1$ does not contain $K_{3,2}$ as a subgraph. Hence, for any subsets $\mathcal{C}'_1 \subset \mathcal{C}_1$ and $\mathcal{V}'_2 \subset \mathcal{V}_2$, we have*

$$I(\mathcal{V}'_2, \mathcal{C}'_1) = O\left(|\mathcal{C}'_1||\mathcal{V}'_2|^{1/2} + |\mathcal{V}'_2|\right).$$

Proof. The first part of the claim is a restatement of Lemma 2.2, and the second part follows from the Kővari–Sós–Túran theorem in extremal graph theory (e.g., see [12, Section 4.5]). \square

Another important observation is that the interaction between red and blue lines is fully symmetric, and the roles of ℓ_1 and ℓ_2 can be interchanged (say, by an appropriate rigid transformation of the plane). We can then associate, in an appropriate parametric plane, hyperbolas with pairs of points of \mathcal{P}_2 , and points with pairs of points of \mathcal{P}_1 , that have exactly the same representations as above (modulo the rigid transformation). In particular, this shows that the incidence graph in $\mathcal{V}_2 \times \mathcal{C}_1$ also does not contain $K_{2,3}$ as a subgraph, and we therefore have:

Corollary 2.4 *For any subsets $\mathcal{C}'_1 \subset \mathcal{C}_1$ and $\mathcal{V}'_2 \subset \mathcal{V}_2$, we have*

$$I(\mathcal{V}'_2, \mathcal{C}'_1) = O\left(|\mathcal{V}'_2||\mathcal{C}'_1|^{1/2} + |\mathcal{C}'_1|\right).$$

Partitioning. We now partition the plane into cells by using a *polynomial partitioning*.² Given a set \mathcal{P} of N points in the plane and a parameter $1 < r \leq N$, we say that $f \in \mathbb{R}[x, y]$ is an *r -partitioning polynomial* for \mathcal{P} if no connected component of $\mathbb{R}^2 \setminus Z(f)$ contains more than N/r points of \mathcal{P} , where $Z(f)$ denotes the zero set of f . Notice that there is no restriction on the number of points of \mathcal{P} that lie in $Z(f)$. The following theorem is a special case of a more general result due to Guth and Katz [9]. A detailed proof can also be found in [11].

Theorem 2.5 (Polynomial partitioning [9]) *Let \mathcal{P} be a set of N points in \mathbb{R}^2 . Then for any $1 < r \leq N$, there exists an r -partitioning polynomial $f \in \mathbb{R}[x, y]$ of degree $O(r^{1/2})$.*

To simplify the notation, we replace the cardinalities of \mathcal{C}_1 and \mathcal{V}_2 by (the larger quantities) n^2 and m^2 , respectively. We construct an r -partitioning polynomial f of \mathcal{V}_2 , where the value of r will be chosen later. Since the degree of f is $O(r^{1/2})$, $Z(f)$ can fully contain $O(r^{1/2})$ hyperbolas of \mathcal{C}_1 . These hyperbolas participate in $O(m^2 r^{1/2})$ incidences with the points of \mathcal{V}_2 (this is a trivial, and rather pessimistic, upper bound).

We next consider incidences between points of \mathcal{V}_2 that are contained in $Z(f)$ and hyperbolas of \mathcal{C}_1 that are not fully contained in $Z(f)$. According to Bézout's theorem (e.g., see [2, Section 8.6]), each hyperbola that is not fully contained in $Z(f)$ intersects $Z(f)$ in $O(r^{1/2})$ points. Thus, the total number of such incidences is at most $|\mathcal{C}_1| \cdot O(r^{1/2}) = O(n^2 r^{1/2})$; this bound subsumes the previous bound.

It remains to bound the number of incidences between the curves of \mathcal{C}_1 and the points of \mathcal{V}_2 that are not contained in $Z(f)$. According to Warren's theorem [16],³ the number of cells in the partition (i.e., the number of connected components of $\mathbb{R}^2 \setminus Z(f)$) is $O(r)$. We denote the number of cells as $t = O(r)$ and the (open) cells themselves as τ_1, \dots, τ_t . Let N_i denote the number of hyperbolas that cross τ_i and let M_i denote the number of points that are contained in τ_i . For every $1 \leq i \leq t$, we have $M_i \leq m^2/r$. By Corollary 2.4, the total number of incidences inside the cells is

$$O\left(\sum_{i=1}^t (M_i N_i^{1/2} + N_i)\right) = O\left(\frac{m^2}{r} \sum_{i=1}^t N_i^{1/2} + \sum_{i=1}^t N_i\right). \quad (7)$$

Each of the hyperbolas $\gamma \in \mathcal{C}_1$ has two connected components, so γ can cross at most two cells of the partition without intersecting $Z(f)$. Every additional cell crossing by γ requires an additional intersection between γ and $Z(f)$. As already argued above, Bézout's theorem implies that there are $O(r^{1/2})$ intersection points between γ and $Z(f)$, so γ crosses $O(r^{1/2})$ cells of the partition. Since this holds for every $\gamma \in \mathcal{C}_1$, we have $\sum_i N_i = O(n^2 r^{1/2})$. According to Hölder's inequality, we have

$$\sum_{i=1}^t N_i^{1/2} = O\left((n^2 r^{1/2})^{1/2} r^{1/2}\right) = O(nr^{3/4}).$$

Combining this with (7) implies that the number of incidences inside the cells is

$$O\left(\frac{m^2}{r} nr^{3/4} + n^2 r^{1/2}\right) = O\left(\frac{m^2 n}{r^{1/4}} + n^2 r^{1/2}\right).$$

²Since the partitioning takes place in the plane, we could also use, equally well, cutting-based partitions; see [12, Chapter 4].

³In the plane, Harnack's theorem [10] can also be used to obtain this bound.

Combining all the partial bounds obtained above, we have

$$I(\mathcal{V}_2, \mathcal{C}_1) = O\left(\frac{m^2 n}{r^{1/4}} + n^2 r^{1/2}\right).$$

We set $r = m^{8/3}/n^{4/3}$, and note that $1 \leq r \leq m^2$ when $m^{1/2} \leq n \leq m^2$. The left inequality holds by assumption. If the right inequality does not hold then we do not construct any partition, and instead apply Corollary 2.4 to obtain $I(\mathcal{V}_2, \mathcal{C}_1) = O(m^2 n + n^2) = O(n^2)$. If r falls in the required range, the bound becomes $O(m^{4/3} n^{4/3})$. Hence we obtain (under the assumption $m \leq n$)

$$I(\mathcal{V}_2, \mathcal{C}_1) = O(m^{4/3} n^{4/3} + n^2).$$

Combining this bound with (1) implies

$$\frac{(mn - x)^2}{x} = O(m^{4/3} n^{4/3} + n^2),$$

or, as is easily checked,

$$x = \Omega\left(\min\left\{m^{2/3} n^{2/3}, m^2\right\}\right).$$

Combining this bound with its symmetric version when $m \geq n$ yields the bound asserted in the theorem. \square

Remarks. (1) As already mentioned in the introduction, closing the (still large) gap between the lower bound of Theorem 1.1 and the near-quadratic upper bound of Elekes [4] remains a major open problem.

(2) Note that $m^{2/3} n^{2/3} = o(m^2)$ when $m = \omega(n^{1/2})$, and then $m^{2/3} n^{2/3} = \omega(n)$. That is, assuming $m \leq n$, the number of distinct distances in the unbalanced case is superlinear when the size m of the smaller set is significantly larger than the square root of the larger size. This fact has recently been established in [14] but without the concrete quantitative bound given in this paper.

(3) We have made use of Elekes's transformation [6] (as refined by Guth and Katz [9]) into an incidence problem between points and lines in \mathbb{R}^3 , mainly in the analysis of the algebraic characterization of quadruples of Q . However, we have completely bypassed the actual machinery of [9], and instead reduced the problem to incidences between points and hyperbolas in the (parametric) plane. As a matter of fact, we did try to combine our approach with the technique of [9], but so far this has not lead to any further improved bound. We leave it as another open problem to improve the bound, if at all possible, by an appropriate mixture of our analysis and the one in [9].

(4) The technique of this paper should also be applicable to distinct distances between sets of points that lie on other kinds of curves. For example, we conjecture that the same lower bound of Theorem 1.1 holds when the sets lie in two non-concentric circles in the plane.

Acknowledgements. The second author would like to thank Andrew Suk for introducing this problem to him.

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